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Topology Change and Context Dependence

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The nonclassical features of quantum mechanics are reproduced using models constructed with a classical theory—general relativity. The inability to define complete initial data consistently and independently of future measurements, nonlocality, and the non-Boolean logical structure are reproduced by these examples. The key feature of the models is the role of topology change. It is the breakdown of causal structure associated with topology change that leads to the apparently nonclassical behavior. For geons, topology change is required to describe the interaction of particles. It is therefore natural to regard topology change as an essential part of the measurement process. This leads to models in which the measurement imposes additional nonredundant boundary conditions. The initial state cannot be described independently of the measurement and there is a causal connection between the measurement and the initial state.

1. INTRODUCTION

It is well known that quantum mechanical systems cannot be described as classical systems evolving in time independently of future measurements (Belinfante, 1973; Beltrametti and Cassinelli, 1981). Attempts to construct classical models either fail or have nonlocal, measurement-dependent, influences. The Kochen–Specker paradox (Belinfante 1973) is an example of the inability to assign initial data (particle spins) independently of future measurements. While in Bohm's theory particle positions and momenta can be defined, they are insufficient to determine the subsequent evolution—the desired results are only obtained at the expense of introducing a nonlocal influence in the form of the pilot wave.

Common to any interpretation of quantum mechanics is the non-Boolean logical structure (Isham, 1968; Beltrametti and Cassinelli, 1981) of the propositions. The propositions have to be defined differently in the different inter-

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pretations (Isham 1968, Section 9.2), so that in some interpretations the propositions are about properties held by the particle, while in other interpretations this would be meaningless and the propositions are statements about the state preparation. The non-Boolean logic of propositions characterizes quantum theory and distinguishes it from classical theory. Once a measurement is defined, and propositions are restricted to those that relate to the chosen measurement apparatus, the results have the familiar logical structure—that this must be the case is explained by Mackey (see Belinfante, 1973; Beltrametti and Cassinelli, 1981, Chapter 13). Unlike any classical theory, quantum mechanics is therefore a context-dependent theory.

It is known that if a breakdown in causal structure is an inherent feature of elementary particles, then the propositions associated with the theory could have the non-Boolean logical structure of quantum mechanics (Hadley, 1997). In this paper models are constructed upon which it is impossible to define complete initial data independently of future measurements. This context dependence is the essence of the Kochen–Specker paradox. Another striking characteristic of quantum theory is the nonlocality predicted, and confirmed, in EPR experiments—this, too, is reproduced in these examples.

As early as 1957 Dennis Sciama pointed out that quantum mechanics was a way to account for hidden variables where half of them were in the future (Sciama, 1957).

A common feature of the models presented in this paper is a breakdown of the causal structure in such a way that there exists a causal link from the measurement to the *initial* data, and that the measurement itself imposes nonredundant boundary conditions. It is known that a theory with these properties will have the logical structure of quantum mechanics. Apart from exotic alternatives (Fivel, 1994), any vector representation will have the familiar Hilbert space structure, operators, and commutation relations.

Models of elementary particles based on geons (where some or all of the particle's properties arise from the topology of spacetime) require topology change to occur when interactions take place. Theorems of Geroch and Tipler place severe constraints on topology change which prevent it occurring without a breakdown of the causal structure in one sense or another. The relationship between topology change and nontrivial causal structure is exploited in this paper. By associating topology change with the measurement process in quantum mechanics, we are naturally led to models which display context dependence.

Geroch's theorem (Geroch, 1967) shows that for a compact time-oriented manifold without closed timelike curves, CTCs, the topology cannot change from one spacelike boundary to the other. Some authors consider the causal structure to be more fundamental and have examined spacetimes with singularities in order to allow topology change (Dowker and Garcia, 1997; Sorkin, 1997). This paper does not consider singularities; it retains the framework of classical general relativity—including a continuous spacetime manifold with a continuous metric. The breakdown in causal structure is actually seen as advantageous because it offers a way of reconciling classical and quantum physics (Hadley, 1997). The examples are created by relaxing the assumptions of Geroch's theorem. We consider in turn manifolds which are not time-orientable, those with CTCs, noncompact manifolds, and finally examples where the spacelike boundaries cannot be defined. In each case the properties are compared with the phenomena previously associated exclusively with quantum theories. If topology change is considered to be an integral part of the measurement process, then these models are examples of context dependence which is the distinguishing characteristic of quantum theory.

2. NOTATION

The paper is concerned with spacetime manifolds of $1 + 1$, $2 + 1$, and $3 + 1$ dimensions with a Lorentzian metric of signature $(-, +)$, $(-, +, +)$, and $(-, +, +, +)$, respectively. The combination of the manifold and the Lorentzian metric is referred to as a *geometry*. The metric defines the concept of a timelike ray (a pair of timelike vectors γ and $-\gamma$ corresponding to a forward and backward direction in time). A geometry which is time-orientable admits a continuous choice of timelike vector γ and is called *isochronous*. An isochronous manifold M therefore admits a continuous vector field. Indeed a manifold which admits a continuous vector field can be endowed with a Lorentzian metric such that it is isochronous (see, for example, Borde, 1997; Reinhart, 1963).

The constructions which follow comprise a geometry M (of dimension $n + 1$) whose boundary is the disjoint union of two manifolds Σ_1 and Σ_2 (each of dimension n). By a topology change we mean that the topology of Σ_1 differs from that of Σ_2. From the results above, a timelike vector field can be constructed in the isochronous case. A unique timelike curve can be constructed through any point such that the curve is everywhere tangent to the vector field. The curves, but not the vector field, can also be constructed in a geometry which is not isochronous.

Most of the paper is concerned with compact geometries. An important class of noncompact spacetimes to which Geroch's theorem and related considerations also apply is made up of externally simple spacetimes (Borde, 1997). A spacetime M is externally simple if there exist compact regions $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$ such that $\Sigma_1 - \hat{\Sigma}_1$ is diffeomorphic to $\Sigma_2 - \hat{\Sigma}_2$.

3. THEOREMS

The following theorem, attributed to C. W. Misner, is a result special to $3 + 1$ dimensions:

Theorem 1 (Misner). Let Σ_1 and Σ_2 be two compact 3-manifolds. Then there exists a compact geometry M whose boundary is the disjoint union of Σ_1 and Σ_2, and in which Σ_1 and Σ_2 are both spacelike.

A proof can be found in Geroch (1967) or Yodzis (1972). The significance of the theorem is that, in the three space and one time dimension of primary interest, a timelike vector field can always be constructed. There are no topological obstructions as there are in other dimensions.

Theorem 2 (Geroch). Let M be a compact 4-manifold with metric signature $(-, +, +, +)$ whose boundary is the disjoint union of two compact spacelike manifolds Σ_1 and Σ_2. Suppose M is isochronous and has no closed timelike curves (CTCs). Then Σ_1 and Σ_2 are diffeomorphic, and further, M is topologically $\Sigma_1 \times [0, 1]$.

Although a 4-manifold was specified (three space and one time dimension) in the original theorem, the proof is also valid for $1 + 1$ and $2 + 1$ dimensions.

Essentially, in order for topology change to take place from one spacelike boundary to another, at least one timelike ray must pass through Σ_1 but not Σ_2 or vice versa. Geroch shows that for compact manifolds there are only two ways for this to happen: (1) The timelike rays that cross the surface Σ_1 and enter M never leave the interior of M, in which case, due to the assumed compactness, there must be a CTC within M; or (2) the timelike rays that cross the surface Σ_1 and enter M, also exit through Σ_1, which implies that M cannot be isochronous.

The first case is further refined by Tipler, who assumes that M is isochronous:

Theorem 3 (Tipler). Topology change cannot occur on an isochronous 4-geometry which satisfies Einstein's equations and the weak energy condition.

Note that Einstein's equations, by themselves, place no constraint whatsoever on the topology (or even the metric). It is only the combination of the equations with some assumptions about physically realistic energy-momentum tensors that places a kinematic constraint.

4. MANIFOLDS WHICH ARE NOT ISOCHRONOUS

When time-orientability is not required, some timelike curves from Σ_1 can enter the manifold and exit through Σ_1 rather than Σ_2. A simple example is $S^1 \to \emptyset$, as depicted in Fig. 1, formed from the cylinder $S^1 \otimes I$, where I is an interval $[0, \tau]$. The first spacelike surface Σ_1 is S^1; in place of the second surface we identify antipodal points to create a Möbius band oriented in the

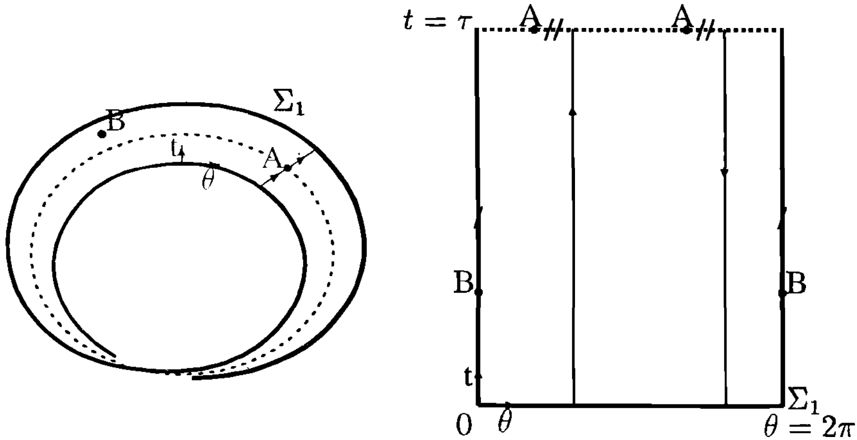

Fig. 1. Topology change from $S^1 \to \emptyset$ to create a manifold that is not isochronous, depicted pictorially and diagrammatically as a rectangle with sides identified. Every timelike curve originating in Σ_1 returns to Σ_1.

space direction, but not the time direction. Every timelike curve from $\Sigma_1(S^1)$ which enters M reemerges through Σ_1 after a finite time.

Initial data on Σ_1 cannot be defined without knowing the value of τ (the time at which the antipodal map occurs). Consider, for example, a wave $\psi(\theta, t)$ which satisfies the one-dimensional wave equation. The general solution $\psi(\theta, t) = \zeta(\theta - vt) + \phi(\theta + vt)$ (with ζ and ϕ having period 2π) is subject to additional constraints; consistency requires $\psi(\theta, t) = \zeta(\theta - vt) + \zeta(\theta + vt - 2v\tau + \pi)$, which depends upon the value of τ.

Clearly the example can be extended to any number of disconnected spaces $n \times S^1 \to m \times S^1$. The model has been described as the annihilation of a universe (Borde, 1997). The main purpose of this paper is to consider kinematics rather than dynamics; however, this particular example is flat—and hence trivially satisfies Einstein's equations in 1 + 1 dimensions. The model can also be extended to higher dimensions. In 2 + 1 dimensions the space orientation is also reversed by the antipodal map. In 3 + 1 dimensions we have $S^3 \to \emptyset$, which satisfies Einstein's equations with a nonzero cosmological constant (it is an Einstein cylinder with the antipodal map imposed). The constraints on initial data in the 3 + 1 case can be demonstrated by considering gravitational waves following null geodesics rather than an unspecified extraneous wave, in which case the manifold is the combination of an underlying slowly varying spacetime and a smaller scale ripple.

If the topology change is associated with a measurement, then the measurements at two different times corresponding to a topology change at $t = \tau_1$ or at $t = \tau_2$ would be incompatible. The topology change cannot take place at both values of τ, and the alternative times would impose different boundary conditions on the problem. To illustrate the non-Boolean logic, we consider either an unspecified one-dimensional wave, a classical object like a billiard ball or (in 3 + 1 dimensions) a gravitational wave that is an intrinsic part of the manifold. In either case, the initial conditions must include a combination of a forward- and backward-moving wave (or particle), and the combination depends upon the value of τ.

Consider a version of the two-slit experiment with slits S_1 and S_2 and a measurement at $t = \tau$ to determine if a billiard ball is in the region $X = [x_1, x_2]$. As described above we associate a topology change and time reversal with the measurement process. The experiment can be carried out with either S_1 or S_2 or both slits open.

With the arrangement of Fig. 2a with only slit S_1 open, there are no trajectories consistent with a measurement at $t = \tau$. Similarly for the case with only S_2 open. With both S_1 and S_2 open, there is now a range of trajectories consistent with a measurement at $t = \tau$ and detection at X, one of which is shown by the dashed line in Fig. 2b.

These examples show that the probability of going through S_1 and reaching X is zero, as is the probability of going through S_2 and reaching X, but the probability of going through (S_1 or S_2) and reaching X is not zero. This is a clear example of non-Boolean logic.

In a technical sense the objectives of this paper have been satisfied by this simple and well-known example of a geometry which exhibits a type of context sensitivity. Initial data cannot be specified independently of the topology change, and there is a clear causal path from the measurement (antipodal

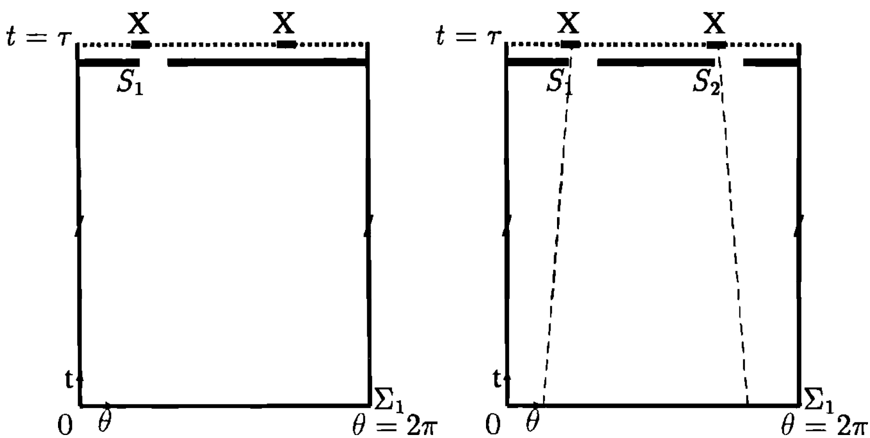

Fig. 2. A two-slit experiment: (a) there are no trajectories consistent with only slit 1 or slit 2 being open, (b) there are consistent trajectories (dashed line) when both slits are open.

map) to the *initial* surface. The parameter τ must be specified before the geometry can be defined.

Unfortunately, continuity prevents this construction being extended to more realistic examples such as a connected boundary Σ_1 changing to a nonempty boundary Σ_2 with a different topology. It is therefore not suitable as a model of particles interacting *within* a universe.

5. SPACETIMES WITH CLOSED TIMELIKE CURVES

Considering compact isochronous geometries, topology change can occur if CTCs are present in the interior of M. Some (or all) curves from the initial surface Σ_1 have no endpoint; they are trapped near a timelike curve. The final surface Σ_2 has some curves which originated in Σ_1 and others that have no past endpoints because they originated from the region of a CTC. There is no limit on topology change in four dimensions if CTCs are allowed (Geroch, 1967).

Trivial examples of topology change with CTCs are known (Borde, 1997; Dowker and Garcia, 1997) in which either the initial or final surface is empty (see, for example, Fig. 3). There are also examples with no timelike curve from the initial to final surface, because they are all confined to the interior of M. In this case Σ_2 cannot be considered to be in the future of Σ_1. These cases will not be considered further because of the lack of any causal connection between the two boundaries.

The examples of interest are those where curves from a subset $A \subset \Sigma_1$ have an endpoint in $B \subset \Sigma_2$ while others are trapped within M. Since Σ_2 is

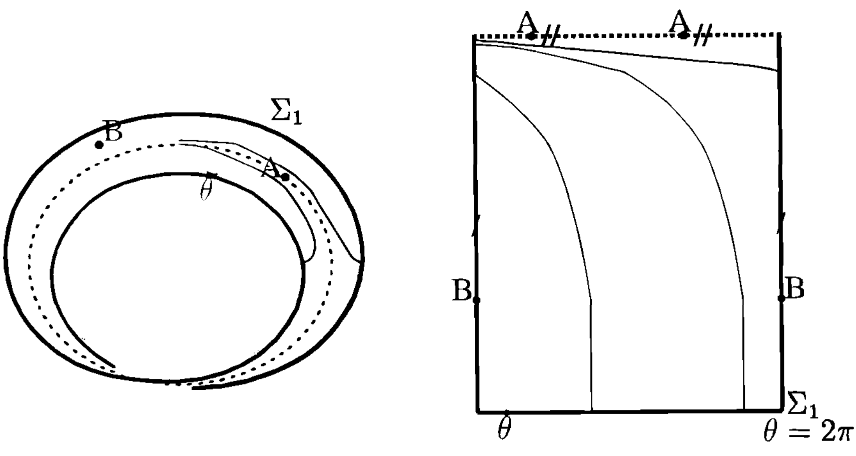

Fig. 3. Topology change $S^1 \to \emptyset$ on a manifold with CTCs. The timelike curves from Σ_1 enter the manifold and never exit. They asymptotically approach the CTC (dashed line).

compact there must be at least one point p of Σ_2 which is not in B. The curve through p cannot have a past endpoint in M and must therefore originate from the region of a CTC.

These geometries do not exhibit quantum phenomena. The topology change introduces unknown (and probably unknowable) boundary conditions because some timelike curves do not originate in the initial surface. However, the topology change places no constraint on the initial data; there is no causal link from the measurement to the initial state nor any clear mechanism for nonlocal effects between separated measurements, as seen in quantum mechanics.

Although there is no dynamical constraint on the formation of CTCs, Tipler's theorem shows that Einstein's equations, plus the weak energy condition, prevents this mechanism for topology change.

6. NONCOMPACT SPACETIMES

Geroch's theorem can be applied to externally simple geometries even when they are not compact because the topology change takes place in a compact region. An externally simple spacetime would be an appropriate description for a geon embedded in $\mathbb{R}^4$. However, continuity still prevents a failure of time-orientability from being a mechanism for topology change.

More generally, topology change can take place in noncompact geometries (Krasnikov 1995) because the timelike curves from Σ_1 can avoid the boundaries without being trapped near a closed timelike curve (see, for example, Fig. 4). These counterexamples can have topology change without a breakdown of the causal structure.

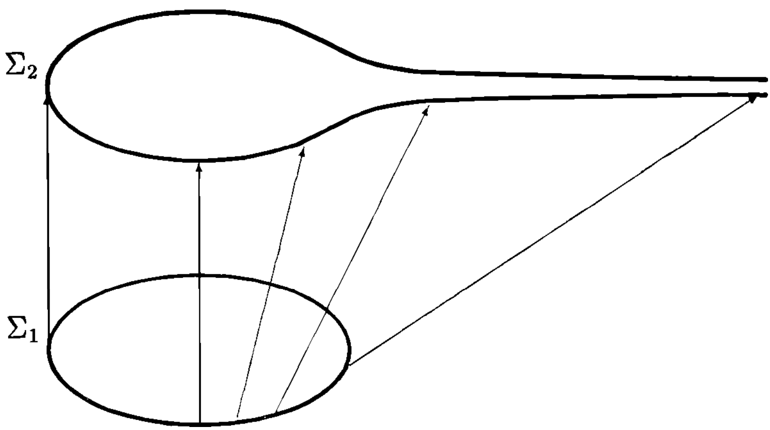

Fig. 4. Topology change $S^1 \rightarrow \mathbb{R}$ on a manifold that is not compact. There is no breakdown of the causal structure.

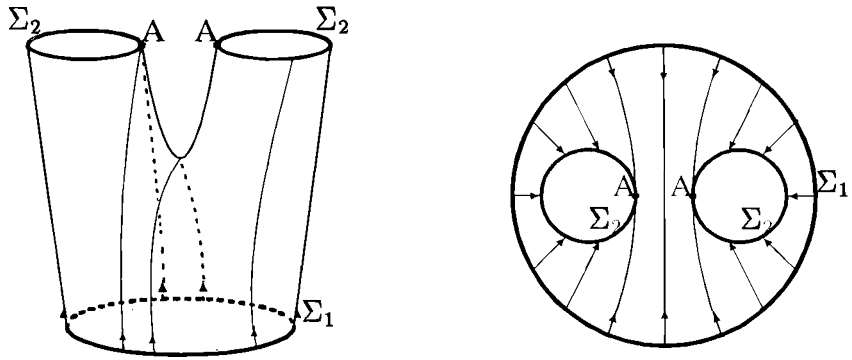

Fig. 5. Topology change from $S^1 \rightarrow S^1 \cup S^1$. Here Σ_1 is spacelike, but some timelike curves from Σ_1 return to Σ_1. The surface Σ_2 is not entirely spacelike. At points A the timelike curves are tangent to Σ_2.

7. NONSPACELIKE BOUNDARIES

A further relaxation of the assumptions in Geroch's theorem is to have boundaries (one or both) that are not entirely spacelike. The terms *initial* and *final* for the boundaries are not strictly correct, although the majority of the surface could be spacelike. With such a significant departure from the conditions of Geroch's theorem, topology change is easily demonstrated. Figures 5 and 6 show a topology change from $S^1 \rightarrow S^1 \cup S^1$; both examples are not time-orientable. Figure 5 has an initial spacelike boundary, but the final boundaries are not spacelike. Figure 6 does not have either an initial or final spacelike boundary, but it is clearly flat, and therefore a trivial solution to the field equations. Another feature of Fig. 6 is the causal link between the two

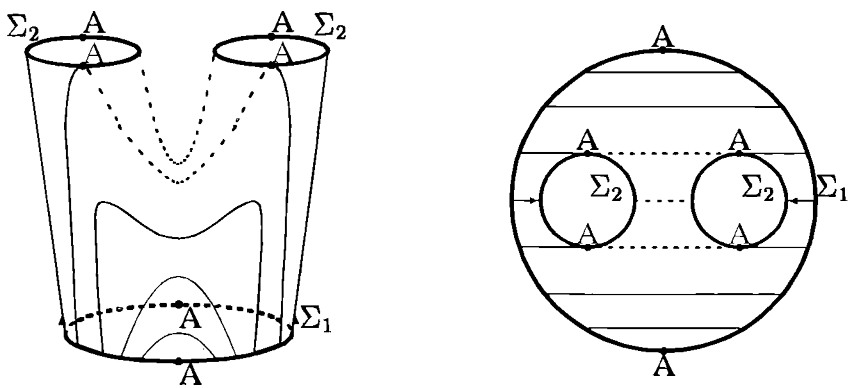

Fig. 6. Topology change from $S^1 \rightarrow S^1 \cup S^1$. The manifold is flat, but neither Σ_1 nor Σ_2 is entirely spacelike. At points A the timelike curves are tangent to the surfaces. Note the timelike curves (dashed curves) joining the different legs of the trousers.

legs of the trousers, which has some comparisons with an EPR experiment. A more realistic model of an EPR experiment would be a combination of three topology-changing regions corresponding to the pair creation, and measurement in the two arms of the experiment.

An externally simple example is shown in Fig. 7, which illustrates $\mathbb{R} \to \mathbb{R} \cup S^1$: the manifold is not isochronous. The mechanism for topology change is essentially that of the nonisochronous case (Section 4); the freedom to allow nonspacelike surfaces permits the construction of nontrivial examples with a continuous metric. In the examples of Figs. 5 and 7, the initial surface Σ_1 is spacelike and time *can* be oriented in a finite neighborhood of Σ_1. It is the subsequent topology change that turns some timelike geodesics back to Σ_1 and obstructs the time-orientability of the geometry.

8. CONCLUSION

In these examples, boundary conditions cannot be specified on the initial surface without knowing about the subsequent measurement (topology change). There is a causal link between the measurement and the initial surface. This breakdown of the causal structure associated with a measurement will necessarily require a context-dependent theory such as quantum theory to describe the observations.

All these compact and externally simple examples prevent, or at least limit, the construction of spacelike boundaries and spacelike hypersurfaces. It must be stressed that this does not indicate any departure from the equations or structure of classical general relativity. To require spacelike boundaries is an extra, convenient, but unwarranted, constraint commonly imposed upon the theory. It is a constraint that is not justified either by the mathematical

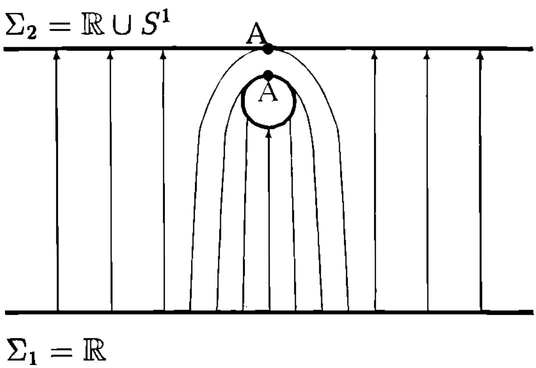

Fig. 7. Topology change from $\mathbb{R} \to S^1 \cup \mathbb{R}$. Here Σ_1 is spacelike, but some timelike curves from Σ_1 return to Σ_1. The surface Σ_2 is not entirely spacelike. At points A the timelike curves are tangent to Σ_2.

structure or by physical reality. Indeed, what we know about quantum theory is incompatible with spacetime being a classical hypersurface evolving with time. Quantum theory gives evidence for the small-scale structure of spacetime, and general relativity could, in return, explain the origins of quantum phenomena.

Of all the interpretations of quantum mechanics, this work relates closely to the pragmatic interpretation of quantum mechanics, where the wavefunction is simply a way of determining the probabilities of the different outcomes from an experiment.

Consider the classical case; $P(n) = 1/6$ $(n = 1, 2, \ldots, 6)$ is a probability function which gives the result of a dice throw, but there is no direct relationship between $P(n)$ and the underlying Newtonian laws of motion. Indeed the probability function does not depend upon the exact form of the equations of motion, it depends only upon the structural form and the symmetry. The classical equations can be expressed as deterministic equations with uniquely defined trajectories determined by the initial conditions alone. The sets of initial conditions corresponding to different results satisfy a Boolean algebra. In principle, a measure on the space of initial conditions can be used to calculate the probability function $P(n)$. The equations of motion are independent of n; this gives the essential symmetry.

With the measurement-dependent effects shown here, general relativity gives a different structure. A measure over *initial* conditions and measurement conditions would also give a Boolean logic as required by Mackey (1963), but a measure over *initial* conditions alone would give a different logical structure which must be orthomodular (Hadley, 1997; Beltrametti and Cassinelli, 1981). The orthomodular structure can be represented by the projections on a Hilbert space (Beltrametti and Cassinelli, 1981). Given the Hilbert space structure, it is well known that the equations for the wavefunction can be derived from symmetry arguments (Ballentine, 1989; Weinberg, 1995). As in the classical case, the probability function does not depend upon the details of the equations of motion, just on their form.

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